EFFICIENT PUBLIC GOOD PROVISION IN NETWORKS: REVISITING THE LINDAHL SOLUTION

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Abstract. The provision of public goods in developing countries is a central challenge. This paper studies a model where each agent's effort provides heterogeneous benefits to the others, inducing a network of opportunities for favor-trading. We focus on a classical efficient benchmark – the Lindahl solution – that can be derived from a bargaining game. Does the optimistic assumption that agents use an efficient mechanism (rather than succumbing to the tragedy of the commons) imply incentives for efficient investment in the technology that is used to produce the public goods? To show that the answer is no in general, we give comparative statics of the Lindahl solution which have natural network interpretations. We then suggest some welfare-improving interventions.

Keywords: Networks, public goods, β-core, repeated games, coalitional deviations, institutions, centrality, Lindahl equilibrium.

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1. Introduction

Questions concerning the provision of public goods are central to development research. A large literature describes the problems of public good provision, especially in rural environments. For example, it has been argued that individuals with similar levels of private consumption but differing levels of public goods such as clean drinking water or effective medical care will experience vastly different qualities of life (Besley and Ghatak [2006]).

Public goods and externalities have been a fundamental topic in economics since Pigou [1920]'s landmark contribution in which he described how taxation can improve welfare. In a later article, Hardin [1997], suggested that public goods such as common lands were overutilised by the community since private individuals did not internalise the social cost when deciding how many cattle to graze. In recent years, Seabright [1993] and Dasgupta [2009] have argued that social institutions and norms have some capacity for aligning private and social incentives in the absence of explicit property rights.[1]

This paper is focused on situations where the provision of public goods (or abatement of public bads) confers multilateral and heterogeneous benefits across players and has heterogeneous costs for each player. We can think of many such examples in a development setting - for example, Foster and Rosenzweig [1995] and Bandiera and Rasul [2006] consider the case of farmers learning about technology as a public good which has positive externalities across farmers.

Consider a typical village. Soil types and wealth levels vary greatly leading to different varieties of crops being grown. A farmer trialing a new hybrid seed would only offer learning advantages to other villagers who would plant the same seed. Also the magnitude of the benefit from learning would be proportional to the amount of land available to each farmer. The ability to observe the information may depend on the geographic and social distance between farmers. This suggests that the benefits will be heterogeneous across farmers depending on the capacity to update given the new information. The model outlined in section 2 is flexible enough to allow these heterogeneous benefits across farmers.

To give a concrete example, suppose farmer A grows only soybeans and wheat, farmer B grows soybeans and rice, and farmer C grows wheat, rice and soybeans. If farmer A trials new soybean seeds, it will lead to benefits to both farmers B and C. But, if farmer C trials

[1]For instance, Dasgupta [2009] describes punishment social punishment within communities "His tribe traditionally practiced a form of punishment that involved spearing the thigh muscle of the errant party. When I asked him what would happen if the party obliged to spear an errant party were to balk at doing so, the young man's reply was that he in turn would have been speared. When I asked him what would happen if the person obliged to spear the latter miscreant were to balk, he replied that he too would have been speared!"

wheat varieties it will only lead to benefits for farmer A. The concept of heterogeneous and multilateral benefits explicitly modelled in the paper may lead to interesting effects. For instance, changes in farmer B's experimentation with rice cultivation may affect farmer C's choice of cultivation of wheat, which in turn may affect farmer A's choice of soybean cultivation - even though farmer A does not directly benefit from farmer B's initial experimentation with rice. This will be explicitly considered in the equilibrium concept.

Water resources in farming villages offer yet another example of public goods in development which have these multilateral and heterogeneous benefits. A shortage of water - a crucial input into farming - will have differing impacts on farmers according to the crop, soil and acreage owned by the farmer. Farmers have access to many water resource tools such as land levelling, irrigation and even choice of crop.

During periods of water shortage, each farmer will have the ability to manage their usage of water at differing costs of implementation (for instance, a farmer on very hard soil will find it very costly to level). There will be both heterogeneous benefits (a farmer with water intensive crops will benefit greatly from a reduction in the water shortage) and multilateral benefits. Ostrom [1992] and Bardhan [2000] have described in detail the heterogeneous effects of water access both at a local and international level.

The above examples demonstrate how public goods can be important in villages and have differential impacts and costs across the village. This leads to two key policy questions this paper is interested in trying to answer:

(1) How will a given intervention in a village change the provision of public goods?
(2) If we wanted to increase the provision of certain public goods, what are the optimal interventions for addressing this?

To answer these fundamental policy questions we need a theory of public good provision in a development setting.

Besley and Ghatak [2006] describe how the collective action problem is difficult to overcome both at a governmental level and a decentralised level. However, they do describe theoretical conditions under which some of the difficulties in public good provision may be mitigated. This can occur in particular, if: (i) interactions are more likely to be repeated since those who refuse can more easily be punished, (ii) information is good so that individuals' actions to assist in public good provision can be observed and (iii) there is a strong social structure that can be used to ostracize individuals or can be used to withdraw other forms of cooperation. These conditions are more likely to be satisfied in close-knit villages.

The framework of this paper will assume theoretical conditions under which public good provision may be successful. Based on such a framework, we will study the levels of public goods provided when the benefits of public goods are heterogeneous across agents and there is a network of benefit flows.

This article studies the Lindahl equilibrium - a classical solution in a public goods game which is on the Pareto frontier. Each agent exerts costly effort to produce a public good which confers heterogeneous benefits to different individuals and the cost of producing these public goods varies across individuals. We use insights from Elliot and Golub [2013a,b] who give a simple characterization of Lindahl equilibrium in terms of a single network centrality condition.[2]

This paper's first contribution characterises how the network architecture affects the amount of investment in the public good by each individual and subsequently solves for each individual's welfare conditional on her place in the network. The paper uses a flexible parametric setting, making the characterization of the Lindahl equilibrium more explicit and interpretable than in Elliot and Golub [2013a].

Second, the paper demonstrates the Lindahl equilibrium is unique. Banerjee et al. [2007] argue that "strong enough coordination mechanisms can make almost *any* group outcome implementable. We believe a micro-founded theory of such coordination is required to make this approach interesting and sharpen its predictive power, and we are not aware of any such theory". We fill this gap in the collective action literature since the Lindahl equilibrium is unique, efficient and has bargaining foundations. Since our equilibrium is unique, we are able to offer comparative statics of how changes in the network architecture affects both public good provision and the welfare of each agent in a Lindahl equilibrium. For example: if one individual's costs of public good provision falls, how does the Lindahl equilibrium change? We are able to answer how and who the policy maker should target to increase certain agents' or groups of agents' welfare. Further,

Third, the paper highlights where public good provision may be suboptimal and suggests policy implications to correct such inefficiencies. The paper focuses on the case where individuals may have suboptimal incentives to reduce their marginal cost of effort. The paper does not explicitly model an investment stage but does offer suggestive evidence for the failure to fully internalise the benefits from cost reductions.

The context of public goods in development is already an extensively studied subject, but groups into two main themes.

Political competition for public goods in the presence of finite resources and differing preferences represents one pillar of past research. Previous papers have shown that benefits

[2]The Lindahl equilibrium enjoys several other robustness properties. As shown in Elliot and Golub [2013a], it is robust to coalitional deviations in a repeated game setting. Elliot and Golub [2013b] extend their characterisation of Lindahl equilibria to demonstrate that it is the unique solution in a repeated game with communication, when marginal (but not inframarginal) costs and benefits are observed.

of public goods do not accrue symmetrically to everyone in a community. This can arise as a result of competition or differing preferences. As Hardin [1997] states, 'successful collective action often entails suppression of another group's interest.' Esteban and Ray [2001] theoretically demonstrate how group size can affect the probability of the successful implementation of a project. Therefore, explicitly modelling the heterogeneous benefits that accrue across individuals is important.

A second consistent theme in the literature is the optimal allocation of decision rights with respect to public good provision. Is it optimal to allocate decision rights centrally at the governmental level - where there may beimperfect information - or decentralise at the local level, where some of the positive externalities across villages may be disregarded? Oates [1972] and Besley and Coate [2003] discuss the importance of decision rights and the respective costs over where to allocate decision rights. This paper adds to this literature by characterising the equilibrium public good provision if the decision rights are decentralised at the village level when there are heterogeneous and multilateral benefits.[3]

Although the paper's main focus is public goods provision, by relabeling certain variables, we can consider positive spillovers within teams. For instance, assume each individual produces a single service which requires different inputs by each agent. The absence of any input will potentially reduce the productivity of the final service and subsequently the value. Hence, the model could be extended to consider the case where each agent provides a service which has heterogeneous benefits across the network, and subsequently characterise how much of the service is provided and what level of services is sustainable.

The remainder of the paper is organized as follows. In section 2, we describe the model and the assumptions. Section 3 provides details on how individual characteristics determine relative utility. Section 4 determines how positive individual shocks affect the network, and section 5 determines how an individual is affected by changes to other individuals. Section 6 concludes.

2. Model

There is a set of agents $N = \{1, 2, ..., n\}$ each of whom simultaneously chooses an an effort $a_i \in \mathbb{R}_{>0}$ at a cost of $c_i f(a_i)$ where we assume $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is a continuously differentiable function such that $f'() > 0$ and $f''(.) > 0$. Therefore, we assume each

[3]The water resource problem described earlier would be a classic question over where to allocate the decision rights. For example if individual villages could decide over whether to build a dam and other water conservation methods, they may ignore the benefits or costs that accrue to neighbouring villages. Whereas if the government were to decide whether to build a dam it may lack the necessary local knowledge over the optimal location of the dam. The insights from the model presented in this paper will attempt to model the extent of the public good provision within the village setting,

agent's cost function is strictly convex in her own effort and there is an individual specific cost $\mathbf{c} \in \mathbb{R}^n_{>0}$ which is possibly heterogeneous across agents.

Each individual potentially benefits from the efforts of his neighbours. In particular we assume an agent benefits from a weighted sum of his peers' efforts. Therefore, each individual's single period utility function $U_i : \mathbb{R}^n_{\geq 0} \to \mathbb{R}$ is:

$$U_i(\mathbf{a_t}) = \sum_j R_{ij} a_{j,t} - c_i f(a_{i,t}) \quad \forall i$$

where $R_{ij} \in \mathbb{R}_{\geq 0}$ for all $i \neq j$ and $R_{ii} = 0$ for all i. The repeated game is one in which each agent takes a possibly random effort $a_{i,t}$ in each of infinitely many discrete periods t and payoffs are given by $u_i = \sum_{t=0}^{\infty} \delta^t U_i(\mathbf{a}_t)$. The game is one of complete and symmetric information.

We do not place any restriction on $\sum_j R_{ij}$ being the same across i; thereby allowing for different individual and total benefits across individuals. We denote the $n \times n$ matrix of benefits as $\mathbf{R}$.[4]

We can think of $\mathbf{R}$ as a network of benefit flows, which is depicted in Figure (1). The thickness of the arrows demonstrates the importance of agent j's effort for agent i's utility (R_{ij}). Figure (1) demonstrates how the benefits can be both heterogenous and multilateral across agents. For instance an increase in agent C's effort would confer benefits to agents A and H, with the benefits A receives greater than that of agent H.

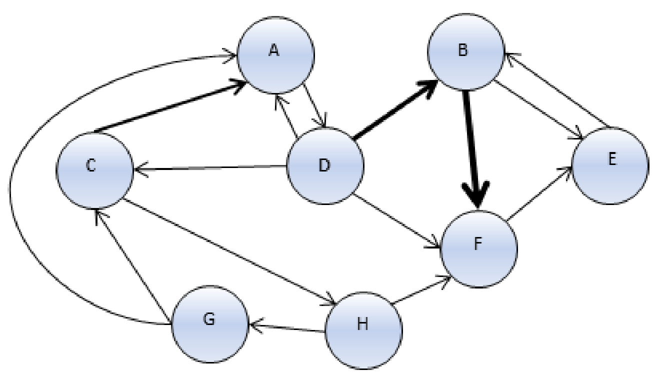

FIGURE 1. The benefit matrix depicted as a network of benefit flows. The thickness of the arrows represents the magnitude of the marginal benefits.

[4]It can be noted that each individual's utility function can be normalised such that $c_i = 1$; however, it will be easier for building intuition for future results to retain the non-normalised utility function.

In the case of farmer learning, we are allowing that some farmers may benefit more from other farmers' experimentation. Additionally, by using different cost parameters, we are allowing it to be more costly for some farmers to invest in learning about new crops - for instance, we may think that learning about new crops is risky, and it is more costly for poorer farmers to insure themselves.

As a technical aside, we note that the above functional form actually represents a more general setting. Consider the following utility function:

$$U_i(\mathbf{a}) = \sum_j R_{ij} h_j(a_j) - g_i(a_i) \quad \forall i \tag{1}$$

Where $h_j(.)$ are continuously differentiable concave functions and $g_i(.)$ are continuously differentiable convex functions. Consider the following transformation of the effort space: $\tilde{a_i} = h_i^{-1}(a_i)$ and $\tilde{g_i}(\tilde{a}) = g_i\left(h_i^{-1}(a_i)\right)$. This transformation maps equation (1) into the basic model with linear benefits, without losing convexity in the cost function. Therefore, the utility function under consideration is both relatively general and tractable.

2.1. **Assumptions on stage game utilities.** A few further technical assumptions are necessary to ensure the existence of a Lindahl (centrality-stable) equilibrium.

Assumption 1 (Sufficiently costly effort). $f'(a) > \max_i \frac{\sum_j R_{ij}}{c_i}$ for some $a \in \mathbb{R}_{>0}$.

Assumption 1 (Sufficiently cheap effort). $f'(a) < \min_i \frac{\sum_j R_{ij}}{c_i}$ for some $a \in \mathbb{R}_{>0}$.

Assumption 3 (Connectedness of the benefit flow). If $\mathbf{a} \in \mathbb{R}^n_{>0}$ and M is a nonempty proper subset of N, then there exist $i \in M$ and $j \notin M$ so that $R_{ij} > 0$

Assumption 1 is required to ensure that we are able to bound the efforts of each agent at an equilibrium effort level.

Assumption 2 is required to ensure that individuals are willing to provide some positive effort.

Assumption 3 is not a particularly restrictive assumption; it is used to ensure that a key matrix is irreducible. If this assumption were to fail, we could conduct the analysis separately on each component.

Key definitions. Following Elliot and Golub [2013a] we define the Jacobian $\mathbf{J}(\mathbf{a})$ at effort levels $\mathbf{a}$ to be the $n \times n$ matrix whose (i, j) element is:

$$J_{ij}(\mathbf{a}) = \frac{\partial U_i}{\partial a_j}(\mathbf{a}) = \begin{cases} R_{ij} & \text{if } i \neq j \\ -c_i f'(a_i) & \text{if } i = j \end{cases}$$

Definition (Elliot and Golub [2013a]). An effort vector $\mathbf{a} \in R^n_{\geq 0}$ is centrality-stable if $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{J}(\mathbf{a})\mathbf{a} = \mathbf{0}$.

The intuition behind this condition is that if we define each individual's contribution as $J_{ii}(a)a_i = -c_i f'(a_i)a_i$ (i.e. the marginal cost of their effort multiplied by the amount of the effort undertaken) this will equal a weighted sum of other agents' efforts, weighted by how much those efforts benefit agent i.

Therefore, at a centrality-stable outcome:

$$\begin{aligned} -J_{ii}(\mathbf{a})a_i &= \sum_j J_{ij}(\mathbf{a})a_j \\ c_i f'(a_i)a_i &= \sum_j R_{ij}a_j \end{aligned}$$

Elliott and Golub demonstrate there is a Lindahl interpretation of this result. They show that in this model, centrality-stable outcomes are exactly the Lindahl equilibria.

2.2. **Lindahl equilibrium.**

Definition 1. A Lindahl equilibrium is a pair $(\mathbf{a},\ \mathbf{P})$, where $\mathbf{a} \in \mathbb{R}^n_{\geq 0}$, and $\mathbf{P}$ is a matrix of prices, such that the following conditions hold:

(i) $P_{ij} \geq 0$ for $i \neq j$ (where P_{ij} can be interpreted as the price i pays for j's effort)

(ii) $\forall j\ P_{jj} = -\sum_{i \neq j} P_{ij}$ (j's wage is the sum of the prices each i pays j)

(iii) $\mathbf{a}^* \in \arg\max \quad U_i(\mathbf{a})$ subject to $\sum_j P_{ij}a_j \leq 0$

Elliot and Golub [2013a] show that $\mathbf{a}$ is a centrality-stable point if and only if there is a matrix of prices ($\mathbf{P}$) such that $(\mathbf{a},\ \mathbf{P})$ is a Lindahl equilibrium. These prices are constructed as follows: $P_{ij} = \gamma_i J_{ij}(\mathbf{a}) = \gamma_i R_{ij}$ for all $i \neq j$, where γ is a row vector such that $\gamma J(\mathbf{a}) = 0$. Therefore, in this context each agent i 'pays' a price to individual j proportional to the marginal benefit (R_{ij}) he receives from individual j's effort. Agent i is 'paid' a price from each farmer in proportion to the marginal benefit (R_{ji}) she receives from agent i's effort.

A further point to note regarding this solution concept: it is robust to coalitional deviations. More precisely, with patient players (sufficiently high $\delta < 1$) it can be enforced by an equilibrium in which, if an agent or coalition of agents does not put in the equilibrium level of effort, then the deviating coalition will not receive any future benefits from those outside the coalition.

Therefore, the centrality-stable equilibrium concept has many appealing features. Not only does it offer a tractable equilibrium condition, it is robust to coalitional deviations, and is able to be sustained in a theoretical Lindahl equilibrium.

Proposition 1. *Under the assumptions in section 2.1, the following statements hold:*

(1) If $\mathbf{a} \in R^n_{>0}$ is centrality-stable, then $\mathbf{a}$ is sustainable.[5]
(2) There exists a centrality-stable $\mathbf{a} \in R^n_{>0}$
(3) If $\mathbf{a} \in R^n_{>0}$ is sustainable, then $\mathbf{a}$ is Pareto-efficient
(4) The centrality-stable $\mathbf{a} \in R^n_{>0}$ is unique and given by:[6]

$$\begin{aligned} \sum_k R_{ik} a_k &= c_i f'(a_i) a_i \ \ \forall i \\ \mathbf{R}\mathbf{a} &= \mathbf{c} \circ \mathbf{f}'(\mathbf{a}) \circ \mathbf{a} \end{aligned} \tag{2}$$

Proof. Statements (1)-(3) hold directly from Elliot and Golub [2013a], and statement (4) is proven using a contraction mapping theorem. The proof is provided in appendix A.1. □

By proving that uniqueness holds in our environment we are able to firstly consider how the equilibrium changes as we alter the parameters of the model. Secondly, as we have proven uniqueness using a contraction mapping theorem we are able to efficiently compute the equilibrium for any functional form which satisfies our assumptions (Stokey et al. [1989]).

3. Individual Characteristics

Understanding how the heterogeneous and multilateral marginal benefits (which can be thought of as a network of benefit flows) affect different outcomes is complex. One way of characterizing the impact of the network is to look at two almost identical agents – clones – and compare how the network and the equilibrium affects their welfare.

Definition 2. Agent i is an *unconnected clone* of agent j if the following conditions hold:

(1) Individuals i and j receive the same marginal benefits from all other agents in the network ($R_{ik} = R_{jk}$ for all $k \neq i$ or j)
(2) Individuals i and j have no reciprocal connections ($R_{ji} = R_{ij} = 0$)

Figure (5) in appendix (B.1) show's an example of identical clones.

Using the definition of clones allows us to examine how the Lindahl equilibrium affects similar individuals in the village.

[5]An effort vector $\mathbf{a}$ is sustainable if there is a $\bar{\delta} < 1$ so that if $\delta > \bar{\delta}$, there is a strong Nash equilibrium σ of the repeated game, which is also a subgame-perfect Nash Equilibrium, in which the infinite repetition of $\mathbf{a}$ occurs on the path of play.

[6]The mathematical operator, $\circ$, is called the Hadamard product. It is a binary operation that takes two matrices of the same dimensions, and produces another matrix where each element ij is the product of elements ij of the original two matrices.

Proposition 2. *Under the assumptions in section 2.1; if individual i is an unconnected clone of agent j and has a higher cost parameter ($c_i > c_j$), then individual i will choose a strictly lower effort ($a_i < a_j$) in the unique Lindahl equilibrium.*

The proof of this result follows from observing that each agent receives the same marginal benefits. Accordingly, the individual with the lower cost parameter will choose the greater effort level (formally shown in appendix A.2) in the unique Lindahl equilibrium.

Definition 3. The elasticity of total cost with respect to effort is defined as:

$$\epsilon(a) = \frac{\partial c_i f(a)}{\partial a} \frac{a}{c_i f(a)} = \frac{\partial f(a)}{\partial a} \frac{a}{f(a)}$$

The elasticity of total cost with respect to effort is the percentage change in total costs divided by the percentage change in effort levels.

If the elasticity of total cost is increasing in the level of effort then the marginal cost $\left(\frac{\partial f(a)}{\partial a}\right)$ divided by the average cost ($\frac{f(a)}{a}$) is increasing in effort.

Proposition 3. *Under the assumptions in section 2.1, if the elasticity of total costs is decreasing in effort and individual i is an unconnected clone of agent j with a higher cost parameter ($c_i > c_j$), then agent i will be better off ($U_i(\mathbf{a}) > U_j(\mathbf{a})$) in the unique Lindahl equilibrium.*

The proof follows from Proposition (2) and the fact that each individual receives the same benefits through the network, therefore the difference in each individual's welfare will be a function of her equilibrium effort level and cost function. The proof is in appendix A.3.

The effect of elasticity decreasing in a can also be shown:

Proposition 4. *Under the assumptions in section 2.1, if the elasticity of total costs is increasing in effort and individual i is an unconnected clone of agent j with a higher cost parameter ($c_i > c_j$), then agent i will be worse off ($U_i(\mathbf{a}) < U_j(\mathbf{a})$) in the unique Lindahl equilibrium.*

Proof. Proof is symmetric to the previous theorem and is omitted. □

In the case where agent i is an unconnected clone of agent j, we can summarize the above results as follows:

$$c_i > c_j \Leftrightarrow a_i < a_j \Leftrightarrow \begin{cases} U_i < U_j & \text{if } \epsilon(a) \text{ is increasing in a} \\ U_i > U_j & \text{if } \epsilon(a) \text{ is decreasing in a} \end{cases}$$

FIGURE 2. How costs, efforts and utilities are related for unconnected clones i and j if $c_i > c_j$ and elasticity is decreasing in a

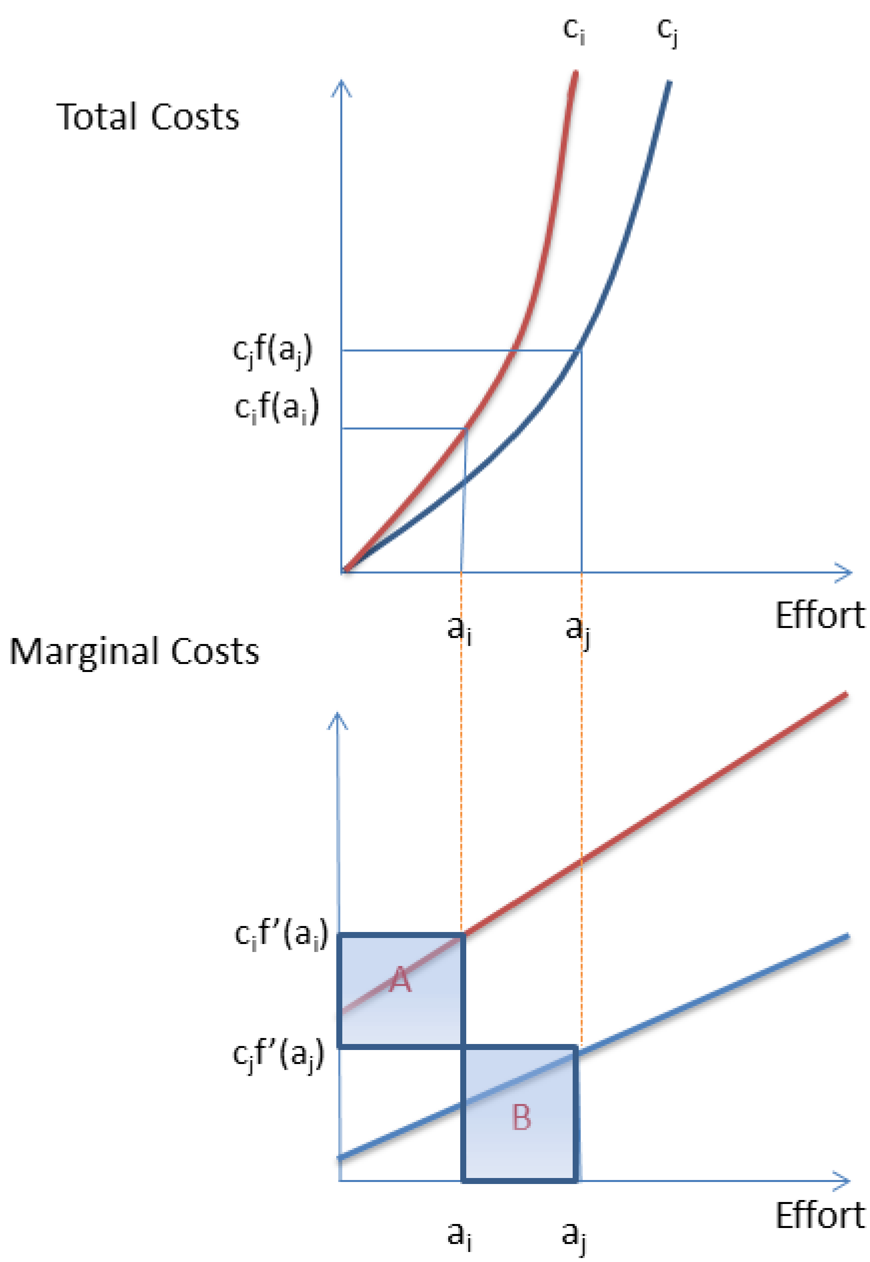

Figure 2 helps to build the intuition for this result. The graph shows a function which exhibits decreasing elasticity in a. Note individual i's cost for a given effort level is always greater than that of agent j. If agent i is an unconnected clone of agent j then the total benefits must be the same for the two individuals. Following from equation (2) this implies:

$$c_i f'(a_i)a_i = c_j f'(a_j)a_j \tag{3}$$

Equation (3) requires that the marginal cost multiplied by the effort level must be equal across unconnected clones in the unique Lindahl equilibrium. Hence area 'A' and area 'B' in figure 2 must be equal at the equilibrium effort levels a_i and a_j. Although agent i has a higher cost parameter c_i, and consequently higher marginal costs at the equilibrium effort a_i, agent i's total costs $(c_i f(a_i))$ are lower than agent j's. Since we have already shown that the total benefits for i and j must be the same, agent i must be strictly better off than agent j in the unique Lindahl equilibrium.

The intuition behind this result stems from the equilibrium condition being determined by marginal costs and benefits. If $\epsilon(a)$ is decreasing in a, the proportional rise in marginal costs is less than that of average costs. It follows that if agent i has a higher cost parameter (c_i) than agent j, then her marginal costs will always be greater than that of agent j. Agent i chooses a lower equilibrium effort a_i which has higher marginal costs - but the proportional rise in marginal costs is less than the rise in average costs. Thus, agent i's total costs are lower and agent i is better off.

Definition 4. Agent i is a *connected clone* of agent j if the following conditions hold[7]:

(1) Individuals i and j receive the same marginal benefits from all other agents in the network ($R_{ik} = R_{jk}$ for all $k \neq i$ or j)
(2) Individuals i and j have equal reciprocal connections ($R_{ji} = R_{ij} = K > 0$)

Figure (6) in appendix (B.2) show's an example of connected clones.

Proposition 5. *Under the assumptions in section 2.1, suppose the elasticity of total costs is decreasing or constant in effort and individual i is a connected clone of agent j with higher individual costs ($c_i > c_j$), then agent i will be better off ($U_i(\mathbf{a}) > U_j(\mathbf{a})$) in the unique Lindahl equilibrium.*

The proof of this result follows from the proposition (3) and is formally provided in appendix A.4. There are two effects when there are reciprocal benefits between agents; (i) the direct effect - person j exerts higher effort than person i, hence person i receives more benefits from his neighborhood and (ii) the effect from the elasticity of the cost function leading to lower total costs. Therefore, the direct effect just enhances the effect mentioned in proposition (3).

[7]The only difference between a connected and unconnected clones is whether there are reciprocal benefits between the clones.

Remark 1. Suppose the elasticity of total costs is increasing in effort and individual i is a connected clone of agent j with higher individual costs ($c_i > c_j$), this does *not* imply that agent i will be worse off ($U_i(\mathbf{a}) < U_j(\mathbf{a})$) in the unique Lindahl equilibrium. There are two effects when agents are connected clones: (i) the direct effect that person j exerts greater effort and (ii) agent i's total costs are higher due to the elasticity increasing in a. Hence, agent i receives more benefits but also incurs higher costs, therefore whether agent i is better off overall is ambiguous.

Propositions (3) - (5) describe how an individual's characteristics affect both her equilibrium effort level and utility. It is not necessarily true that individuals who have cheaper technology are better off and in particular it depends on the technology environment. Therefore, if we have two different farmers who receive the same benefit from their neighbours' investments, the farmer for whom it is more costly to invest in learning about new methods may be better off.

Definition 5. Agent i is an *identical clone* of agent j if the following conditions hold[8]:

(1) The cost parameter for individuals i and j are the same: $c_i = c_j = c$
(2) Individuals i and j receive the same marginal benefits from all other agents in the network ($R_{ik} = R_{jk}$ for all $k \neq i$ or j)
(3) Individuals i and j confer the same marginal benefits to all other agents in the network ($R_{ki} = R_{kj}$ for all $k \neq i$ or j)
(4) Individuals i and j have equal reciprocal connections ($R_{ji} = R_{ij} = K \geq 0$)

Figure (7) in appendix (B.3) show's an example of identical clones.

Corollary 1. *Under the assumptions in section 2.1, suppose the elasticity of total costs is decreasing in effort and individual i is an identical clone of agent j. If agent i's cost was to fall then agent j would be better off than agent i ($U_j(\mathbf{a}) > U_i(\mathbf{a})$) in the unique Lindahl equilibrium.*

The proof follows from propositions (3) and (5). If the elasticity of total costs is decreasing in effort, we know from propositions (3) and (5) that the agent with higher costs will be better off. When the agents have the same cost and confer the same marginal benefits to all other agents in the network we also know that a reduction in either of the cost functions will lead to the same equilibrium effort vector ($\mathbf{a}$) in the unique Lindahl equilibrium. Therefore, if agent i could choose between reducing her own cost or farmer j's cost, she would always choose to reduce farmer j's cost.[9]

[8]Therefore identical clones differ from the connected and unconnected clones in that they have the same costs and confer the same marginal benefits to all other agents.

[9]We have had to make the additional assumptions 'individuals i and j confer the same marginal benefits to all other agents in the network ($R_{ki} = R_{kj}$ for all $k \neq i$ or j)' and '$c_i = c_j$' in our result because although

Consequently, farmer i has a greater incentive to reduce c_j than her own cost parameter c_i. This is a startling result and follows from: when the elasticity of total costs is decreasing in effort for a given effort level of their neighbours, each farmer would prefer to have a higher cost parameter. We have not modelled a two stage game, therefore, this result is more suggestive of how even if we have Pareto optimality in the stage game, initial incentives to invest in cost reducing technology may be distorted.

4. Network-wide Characteristics

The paper can also examine how changes in the network can affect individual effort levels and welfare. Firstly a result pinning down the indirect utility function for each agent will be useful for explaining the intuition and the proofs.

Definition 6. An agent's indirect utility function can be characterised by $U_i(c_i,\ a_i^*(\mathbf{R},\mathbf{c})) = U_i(\mathbf{R},\mathbf{c})$ such that $\mathbf{Ra} = \mathbf{cf}(\mathbf{a})$

Lemma 1. *An agent's indirect* $U_i(c_i,\ a_i(\mathbf{R},\mathbf{c}))$ *is strictly increasing in the equilibrium* a_i

The proof is provided in appendix A.5. Ultimately, if an farmer's effort level increases and c_i stays the same, then it must be because farmer i's benefits have increased. The increase from the benefits will always be greater than the rise in costs.

Definition 7. Define the cost vector $\mathbf{c}$ to be less than $\mathbf{c}'$, $\mathbf{c} < \mathbf{c}'$, if $c_i \leq c_i'$ for all i and $c_j < c_j'$ for at least some one j.

Definition 8. Define the matrix $\mathbf{D}(\mathbf{a})$ to be a positive diagonal matrix such that D_{ii} is the marginal cost for agent i to increase her effort level at the equilibrium action $\mathbf{a}$: $D_{ii}(\mathbf{a}) = \frac{\partial}{\partial a_i}[c_i f'(a_i)a_i] = c_i f''(a_i)a_i + c_i f'(a_i)$.

Definition 9. Define the matrix $\mathbf{M}(\mathbf{a})$ to be the weighted distance between individual i and j at the equilibrium effort level.[10] Element $M_{ij}(\mathbf{a})$ is a weighted sum of the number of paths between agent i and j, weighted by the marginal benefit (R_{ij}) of an increase in a_j for individual i divided by the marginal cost for agent j to increase his effort (D_{jj}). Therefore element M_{ij} is useful to describe how important individual j is for i taking into account all possible paths between i and j in the unique Lindahl equilibrium.

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we know that the farmer with higher costs would be better off than the farmer with lower costs, we do not know how the other individuals effort levels are affected in equilibrium. Therefore by assuming the two additional assumptions above we can explicitly say that regardless of whose costs are reduced everyone elses' equilibrium effort level would be the same.

[10]Formally $\mathbf{M}(\mathbf{a}) \equiv (\mathbf{I_n} - \mathbf{RD}(\mathbf{a})^{\mathbf{-1}})^{-1}$ and it is shown in Appendix (A.6) this can be written as $\mathbf{M}(\mathbf{a}) \equiv \left[\sum_{k=0}^{\infty} \left(\mathbf{RD}(\mathbf{a})^{\mathbf{-1}}\right)^k\right]$

Proposition 6. *Under the assumptions in section 2.1, an increase in the cost vector* $\mathbf{c}' > \mathbf{c}$ *will necessarily lead to a decrease in the effort levels for all agents at the the unique Lindahl equilibrium. In particular each agent's change in effort level for a small change in the cost vector is equal to:*

$$(\mathbf{a}' - \mathbf{a}) \simeq -\mathbf{D}^{-1}(\mathbf{a})\mathbf{M}(\mathbf{a})\,(\mathbf{c}' - \mathbf{c}) \circ \mathbf{f}'(\mathbf{a}) \circ \mathbf{a}$$

The proof is provided in appendix A.6.

In equilibrium, we would expect that an individual's action would fall as we increase his cost function. However, it is not immediately obvious that an increase in one individual's cost will necessarily lead to a reduction in everyone's effort in equilibrium.

Intuitively, if farmer i's costs rise, for a given effort vector of his peers $\mathbf{a}_{-i}$, farmer i will reduce his effort level.[11] If farmer i reduces his effort, any farmer j who receives benefits directly from farmer i ($R_{ji} > 0$) will reduce his effort since his benefits are reduced. We can now extend the argument to all farmers k who receive benefits from the set of farmers j who are directly connected to farmer i. Repeating this chain we can see all farmers reduce their effort levels.

Due to the network of benefit flows, those farmers who confer benefits to farmer i will also reduce their effort levels. Subsequently, there is a multiplier effect from farmer i's initial reduction in effort which flows through the network (the magnitude of this multiplier effect for each agent depends on the matrix $\mathbf{M}$). Figures (3) and (4) graphically demonstrate the intution. Figure (3) shows the initial stylised network of benefit flows and figure (4) shows how changes in person G's cost parameter will cause changes through the network. Assume that G's cost parameter was to rise, for a given effort vector for the other agents, $G's$ effort will fall. Therefore, the benefits A recieves will fall, leading A to reduce her effort. Following the chain, we know that F will reduce his effort, but then G will reduce her effort further, leading to a new iteration of effort reductions through the network (the multiplier effect).[12]

Overall, farmer i reduces her equilibrium effort due to two effects: (i) the direct cost effect and (ii) the indirect effect from other farmers reducing their own effort levels leading to lower benefits for farmer i.

Corollary 2. *Under the assumptions in section 2.1, a decrease in the cost vector* $\mathbf{c}' < \mathbf{c}$ *will necessarily lead to an increase in the effort levels for all agents in the unique Lindahl equilibrium.*

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[11]This follows from proposition (2).

[12]In the appendix, we show that the spectral radius of $\mathbf{RD}^{-1}$, $\rho(\mathbf{RD^{-1}})$, is less than one. Therefore, we can conclude the effect of the cost increase is a converging series.

FIGURE 3. A stylised benefit flow through the network. The arrows demonstrate who confers benefits on whom.

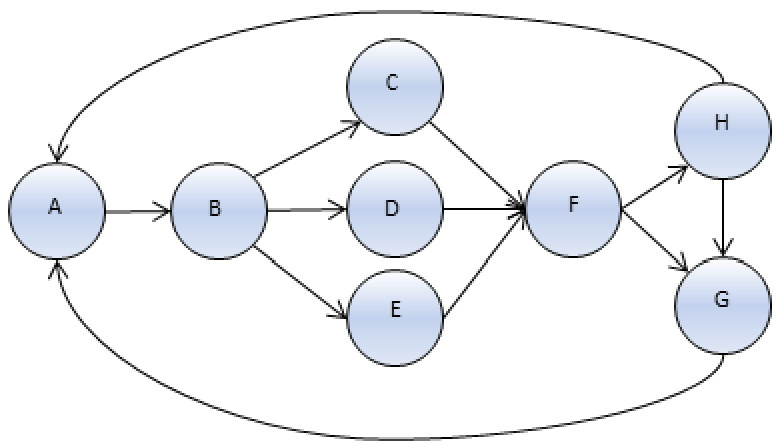
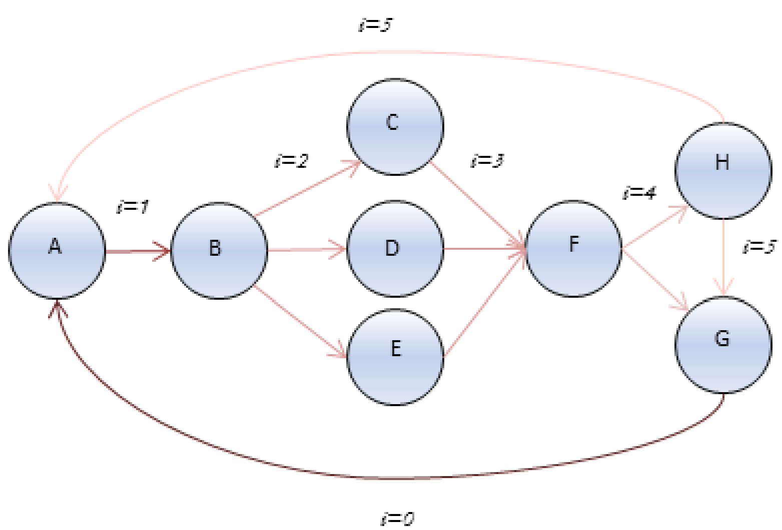


FIGURE 4. The flow of a cost increase on person G through the network



Proof. The proof is symmetric to the previous proposition and is omitted □

Definition 10. The matrix of benefits $\mathbf{R}'$ is greater than $\mathbf{R}$ ($\mathbf{R}' > \mathbf{R}$), if $R'_{ij} \geq R_{ij}$ for all ij pairs and $R'_{ij} > R_{ij}$ for at least one ij pair.

Proposition 7. *Under the assumptions in section 2.1, an increase in* $\mathbf{R}'$ *from* $\mathbf{R}$ *such that* $\mathbf{R}' > \mathbf{R}$ *will necessarily lead to an increase in the effort levels for all agents in the unique Lindahl equilibrium.*

The proof is provided in the appendix A.7.

Similar to the intuition for Proposition (6), if farmer i has an increase in his benefit matrix, then farmer i's effort level will increase. Therefore, those farmers who receiver benefits from farmer i will also increase their equilibrium effort level. Consequently, a multiplier effect through the network will lead to each agent increasing their equilibrium action in the unique Lindahl equilibrium.

By combining proposition (7) with Lemma (1), we can see an increase in the benefit matrix for any agent will necessarily lead to *the increase in utility for all agents* - explicitly, the benefits for farmer i from other agents increasing their effort levels outweighs the costs for farmer i increasing her own effort level.

In terms of policy advice, attempting to increase the proliferation of public good provision within a village may be supported by investing in improving network links between agents. For instance, if new hybrid seeds were invented which were sustainable over many different soil types, this would increase the benefit matrix, which would lead to greater crop experimentation.

Furthermore, we expect that it is easier to share information in villages which have a greater amount of social links between farmers, therefore those vilages which have greater social intergration may lead to a higher level of public good provision since the benefits from investing would be greater. This suggests improving village institutions to share information may improve each farmer's welfare.

One further point; in this model we consider the benefit matrix to be exogenous, however, if each individual were to privately invest in these links, there would be a suboptimal level of link formation. Since every farmer's utility increases as we improve a link between farmers i and j, this means there is a positive externality from the introduction of each link on the rest of the village. Hence, the social benefits are greater than the private benefits from link formation leading to the possibility of inefficient link formation. This further supports the possibility that even if we have Pareto efficient actions being undertaken, there may be inefficiencies in the first stage, whether through insufficient incentives to form links or implementing cost reducing technology.

Proposition 8. *Under the assumptions in section 2.1, an increase in the cost vector from* $\mathbf{c}$ *to* $\mathbf{c}'$ *such that* $\mathbf{c} > \mathbf{c}'$ *will necessarily lead to a reduction in utility for all agents at the unique Lindahl equilibrium.*

The proof is provided in appendix A.8.

This proof shows that even though everyone's equilibrium action falls (so the total costs for each individual falls), the reduction in costs is less than the reduction in benefits for each individual. Secondly, it demonstrates no agent has an incentive to increase their own costs.

Summarising, propositions (3) and (5) argue an individual with higher costs may be better off than an individual with lower costs and proposition (8) argues that if either of their costs were reduced both individuals would be better off.

5. How the network affects the individual

In this section we will be analysing how each individual's welfare is a function of the entire network. Recall the matrix (definition 9) $\mathbf{M}(\mathbf{a})$ introduced in section 4. Element $M_{ij}(\mathbf{a})$ is useful to describe how important individual j is for i taking into account all possible paths between i and j.

Proposition 9. *Under the assumptions in section 2.1, consider an increase in costs for person* k*, then the relative impact on person* i *and person* j *can be characterised by individual fixed effects* ($H_i(a_i)$ *and* ($H_j(a_j)$)*, and network effects from person* k *to person* i *and* j (($M_{ik}(\mathbf{a})$) *and* ($M_{jk}(\mathbf{a})$)) *in the unique Lindahl equilibrium.*

$$\frac{\frac{dU_i}{dc_k}}{\frac{dU_j}{dc_k}} = \frac{H_i(a_i)}{H_j(a_j)} \times \frac{M_{ik}(\mathbf{a})}{M_{jk}(\mathbf{a})} \quad \textit{for } k \neq i \quad \textit{or } j$$

where $H_i(a_i) \equiv \frac{f''(a_i)a_i}{f''(a_i)a_i + f'(a_i)}$

The proof is in appendix A.9.

Therefore, this proposition formally shows how each agent's relative welfare is affected by a change in farmer k's costs. Furthermore, if person i and j were exerting the same effort initially, the farmer's whose utility would be most affected would be the farmer who is 'closest' to farmer k in terms of weighted paths.

Proposition 10. *Under the assumptions in section 2.1, consider a change in the benefit matrix* (R_{kj}) *then the change in person* i*'s welfare can be characterised by an individual fixed effect* ($H_i(a_i)$)*, a network effect from person* k *to person* i ($M_{ik}(\mathbf{a})$) *and the action of person* j *in the unique Lindahl equilibrium.*

Proof. The change in person i's utility is equal to:

$$\begin{aligned} \frac{dU_i}{dR_{kj}} &= \frac{c_i\left(f''(a_i)a_i\right)}{c_i\left(f''(a_i)a_i + f'(a_i)\right)} M_{ik}(\mathbf{a})a_j \quad \text{if } k \neq j \\ &= H_i(a_i) \times M_{ik}(\mathbf{a}) \times a_j \quad \text{if } k \neq j \end{aligned} \tag{4}$$

□

Therefore, if were to consider two different possible link improvements R_{kj} or R_{kp} and compare which link addition would benefit farmer i the most, we can observe this is solely dependent on whether farmer j or farmer p puts in the most effort since:

$$\frac{\frac{dU_i}{dR_{kj}}}{\frac{dU_i}{dR_{kp}}} = \frac{a_j}{a_p} \quad \text{for } k \neq j \quad \text{or } p \tag{5}$$

Intuitively, this follows from observing when we improve a link to agent k, agent k's direct benefit is largest when the link comes from the agent who exerts the highest effort. Since the multiplier effect described in section 4 is proportional to the increase in agent k's effort, agent i prefers for the link to farmer k to induce the greatest rise in his effort and consequently the network.

We can extend this result a little further. Consider the following thought exercise: If we were to improve one link in the network from any agent k to any agent j such that $k \neq j$, which link should we improve to maximise farmer $i's$ utility? Equation (4) shows that we only to look at two different variables, the action vector ($\mathbf{a}$) and the distance between the recipient of the benefit and farmer i (M_{ij}). We conjecture under quite general conditions that $M_{ii} > M_{ik}$ for all $k \neq i$. In this case, if we wanted to maximise farmer i's utility from improving a benefit link, we should add a benefit link from the farmer j who has the highest effort level to farmer i, where $j \neq i$.

6. Conclusion

Close-knit villages are characterised by repeated interaction, and a large, complex social network which provides information and potential social sanctions. These three key features of village economies suggests that a cooperative equilibrium which is coalitionally stable may occur. The Lindahl equilibrium in our model is a useful benchmark for these villages for four reasons: it is unique, robust to coalitional deviations, has strategic foundations, and offers a market interpretation whilst simultaneously being characterised by a single network centrality condition. The solution concept has many desirable properties whilst offering a tractable model for analysing the provision of public goods.

This article examines three key features of the network: (i) how an individual's relative welfare in the network depends on their own characteristics, (ii) how individual changes affect the network and (iii) how individual's welfare changes according to the network architecture.

The paper demonstrates how the provision of public goods within a network can change with small changes in the parameters. Additionally, the paper is able to characterise the welfare effects from these changes in the environment.

Although not explicitly modelled, the paper also shows how the optimal provision of public goods may fail. In particular, by showing a cost reduction for agent i or an improvement in the benefit matrix has positive externalities for all other individuals in the network suggest there are suboptimal incentives for investment in cost reducing technology. Future research focusing on explicitly modelling the incentives for investment in cost reducing technologies would be interesting.

The paper has many further possible extensions. The utility function in this paper is a weighted sum over other individual's actions, therefore there is no complementarity or substitution between different individuals provision of public goods. For instance, we may expect that learning about different crops are substitutes since farmers can only grow a finite number of crops. Understanding how to increase the flexibility of the model whilst still retaining tractability and ascertaining whether uniqueness still holds would be an interesting line for future research.

Another interesting question for future research arises from ascertaining what is the welfare maximising benefit matrix? Assume that there is a fixed number of benefit links and all agents had the same cost parameter, what would be the optimal configuration? We conjecture that due to the strict convexity in the cost function, the optimal configuration would have an equal number of benefit flows for each agent.

Overall, this paper has described how heterogeneous and multilateral benefits affect the equilibrium provision of public goods under assumptions which are likely to be fulfilled in a close-knit villages. We have characterised the Lindahl equilibrium and described how the equilibrium changes under certain conditions.

References

O. Bandiera and I. Rasul. Social networks and technology adoption in northern mozambique. *The Economic Journal*, 116(514):869–902, 2006.

A. Banerjee, L. Iyer, and R. Somanathan. Public action for public goods. *Handbook of development economics*, 4:3117–3154, 2007.

P. Bardhan. Irrigation and cooperation: An empirical analysis of 48 irrigation communities in south india. *Economic Development and cultural change*, 48(4):847–865, 2000.

T. Besley and S. Coate. Centralized versus decentralized provision of local public goods: a political economy approach. *Journal of public economics*, 87(12):2611–2637, 2003.

T. Besley and M. Ghatak. *Public goods and economic development*. Oxford University Press, 2006.

P. Dasgupta. Trust and cooperation among economic agents. *Philosophical Transactions of the Royal Society B: Biological Sciences*, 364(1533):3301–3309, 2009.

M. Elliot and B. Golub. Network centrality approach to coalitional stability. *Working paper*, 2013a.

M. Elliot and B. Golub. A unique efficient public goods equilibrium robust to incomplete information. *working paper*, 2013b.

J. Esteban and D. Ray. Collective action and the group size paradox. In *American Political Science Association*, volume 95, pages 663–672. Cambridge Univ Press, 2001.

A. D. Foster and M. R. Rosenzweig. Learning by doing and learning from others: Human capital and technical change in agriculture. *Journal of political Economy*, pages 1176–1209, 1995.

R. Hardin. *One for all: The logic of group conflict*. Princeton University Press, 1997.

W. Oates. *Fiscal Federalism*. Harcourt Brace, New York, 1972.

E. Ostrom. Crafting institutions for self-governing irrigation systems. 1992.

A. Pigou. *The economics of welfare*. MacMillan, London, 1920.

P. Seabright. Managing local commons: theoretical issues in incentive design. *The Journal of Economic Perspectives*, pages 113–134, 1993.

N. Stokey, R. Lucas, and E. Prescott. *Recursive Methods in Economic Dynamics*. Harvard University Press, 1989.

Appendix A. Proofs

A.1. **Proof of Proposition 1.** Under the assumptions in Section 2.1, the following statements hold:

(1) If $\mathbf{a} \in R^n_{>0}$ is centrality-stable, then $\mathbf{a}$ is sustainable.
(2) There exists a centrality-stable $\mathbf{a} \in R^n_{>0}$.
(3) If $\mathbf{a} \in R^n_{>0}$ is sustainable, then $\mathbf{a}$ is Pareto-efficient.
(4) The centrality-stable $\mathbf{a} \in R^n_{>0}$ is unique and given by:

$$\sum_k R_{ik} a_k = c_i f'(a_i) a_i \;\; \forall i \tag{6}$$

$$\mathbf{R}\mathbf{a} = \mathbf{c} \circ \mathbf{f}'(\mathbf{a}) \circ \mathbf{a} \tag{7}$$

Proof. Statements (1)-(3) hold directly from Elliot and Golub [2013a], and statement (4) is proven using a contraction mapping theorem.

We want to show there is only one solution to equation (6). If we can show that the following system of equations is a contraction mapping, then we will have a unique solution:

$$\mathbf{g}(\mathbf{a}) = \mathbf{a} - \mathbf{B}\varphi(\mathbf{a}) \tag{8}$$

where $\mathbf{B}$ is some $n \times n$ matrix with constant coefficients to be chosen and $\varphi(\mathbf{a}) \equiv \mathbf{c} \circ \mathbf{f}'(\mathbf{a}) \circ \mathbf{a} - \mathbf{R}\mathbf{a}$. Therefore, the equation $\mathbf{g}(\mathbf{a}) = \mathbf{a}$ is equivalent to $\varphi(\mathbf{a}) = \mathbf{0}$.

Formally if $g: \; X \to X$ is Lipschitz continuous with Lipschitz constant $L < 1$, then $\mathbf{g}(\mathbf{a})$ has a unique fixed point $\mathbf{a}^*$ and thus $\varphi(\mathbf{a}^*) = 0$. Therefore, we must first show that we can find a mapping g from a set X to itself. Then we must show this mapping is a contraction on X.

Let us assume a_i is bounded above by $\bar{a}$ such that $f'(\bar{a}) \equiv \max_i \frac{\sum_j R_{ij}}{c_i} = \bar{K}$ and bounded below by $\underline{a}$ such that $f'(\underline{a}) \equiv \min_i \frac{\sum_j R_{ij}}{c_i} = \underline{K}$. Next let us define the set X to be T^n, where $T = [\underline{a}, \; \bar{a}]$. Also, let us assume our matrix $\mathbf{B}$ is a diagonal matrix with strictly positve elements, b_i. To show that our mapping g, maps set T^n into itself, we must show g satisfies:

$$\bar{a} \geq g_i(\mathbf{a}) \geq \underline{a} \quad \forall \mathbf{a} \in T^n, \; i$$

Where we denote the i^{th} element of $g(\mathbf{a})$, as $g_i(\mathbf{a})$.

First we show $g_i(\mathbf{a})$ is strictly less than or equal to $\bar{a}$ for all $\mathbf{a}$. This condition, element-wise requires:

$$\bar{a} \geq a_i - b_i \left[c_i f'(a_i) a_i - \sum_j R_{ij} a_j \right]$$

Let us define $\bar{R} \equiv \max_j \sum R_{ij}$, then it must hold that:

$$\begin{aligned} \bar{a} &\geq a_i - b_i \left[c_i f'(a_i) a_i - \sum_j R_{ij} a_j \right] \\ &\geq a_i - b_i \left[c_i f'(a_i) a_i - \bar{R}\bar{a} \right] \end{aligned} \tag{9}$$

Since inequality (9) must hold for all possible values of a_i, let us define $\hat{a}_i$ such that:

$$\hat{a}_i = \arg\max_{a_i \in T} a_i - b_i \left[c_i f'(a_i) a_i - \bar{R}\bar{a} \right]$$

If $\hat{a}_i$ does not equal $\bar{a}$, then $\hat{a}$ solves the following implicit function:

$$1 - b_i c_i f''(\hat{a}_i)\hat{a}_i - b_i c_i f'(\hat{a}_i) = 0 \tag{10}$$

Let us first show the case where $\hat{a}_i$ is stricly less than $\bar{a}$. Substituting equation (10) into equation (9) we have:

$$\bar{a} \geq \hat{a} + b_i \bar{R}\bar{a} - \left[1 - b_i c_i f''(\hat{a}_i)\hat{a}_i \right] \hat{a}_i$$

Rearranging and using $\hat{a}_i$ is stricly less than $\bar{a}$, it follows:

$$1 \geq b_i(\bar{R} + c_i f''(\hat{a}_i)\hat{a}_i)$$

Since $c_i f''(\hat{a}_i)\hat{a}_i$ is finite, we can always find a sufficiently small b_i such that this condition holds.

If $\hat{a}_i$ equals $\bar{a}_i$ then inequality (9) becomes:

$$\begin{aligned} \bar{a} &\geq \bar{a} - b_i \left[c_i f'(\bar{a})\bar{a} - \bar{R}\bar{a} \right] \\ 1 &\geq 1 - b_i \left[c_i f'(\bar{a}) - \bar{R} \right] \\ c_i f'(\bar{a}) &\geq \bar{R} \end{aligned}$$

This condition is satisfied by our definition that $f'(\bar{a}) \equiv \max_i \frac{\sum_j R_{ij}}{c_i}$. Therefore we have established $g_i(\mathbf{a})$ is strictly less than or equal to $\bar{a}$ for all $\mathbf{a}$. Next we show that $g_i(\mathbf{a})$ is strictly greater than or equal to $\underline{a}$ for all $\mathbf{a}$. This condition, element-wise requires:

$$\underline{a} \leq a_i - b_i \left[c_i f'(a_i) a_i - \sum_j R_{ij} a_j \right] \tag{11}$$

Let us define $\underline{R} \equiv \min_j \sum R_{ij}$, then it must hold that:

$$a_i - b_i \left[c_i f'(a_i)a_i - \underline{Ra}\right] \leq a_i - b_i \left[c_i f'(a_i)a_i - \sum_j R_{ij}a_j\right] \tag{12}$$

Where we substitite $\left(\underline{Ra} \leq \sum_j R_{ij}a_j\right)$ into equation (11). The LHS of equation 12 is minimized at $a_i = \underline{a}$, therefore we have to merely show that:

$$\underline{a} \leq \underline{a} - b_i \left[c_i f'(\underline{a})\underline{a} - \underline{Ra}\right]$$

Rearranging:

$$c_i f'(\underline{a}) \leq \underline{R}$$

Which is satisfied by our initial assumption that $f'(\underline{a}) \leq \min_i \frac{\sum R_{ij}}{c_i}$. Therefore, we have shown for a sufficiently small b_i, the mapping g, maps from T^n into itself. Now we just need to show that g is a contraction on T^n.

Differentiating equation (8) with respect to $\mathbf{a}$:

$$\mathbf{g}'(\mathbf{a}) = \mathbf{I} - \mathbf{B}\varphi'(\mathbf{a})$$

Note:

$$\varphi'(\mathbf{a}) = \begin{bmatrix} c_1 f''(a_1)a_1 + c_1 f'(a_1) & -R_{12} & \cdots & -R_{1n} \\ -R_{21} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -R_{n-1,n} \\ -R_{n1} & \cdots & -R_{n,n-1} & c_n f''(a_n)a_n + c_n f'(a_n) \end{bmatrix}$$

Therefore, if we can find a suitable matrix $\mathbf{B}$ such that:

$$\| \mathbf{I} - \mathbf{B}\varphi'(\mathbf{a}) \| < 1 \tag{13}$$

then we have contraction mapping.

Then we can write equation $\mathbf{I} - \mathbf{B}\varphi'(\mathbf{a})$ element wise as:

$$\begin{bmatrix} 1 - b_1\left(c_1 f''(a_1)a_1 + c_1 f'(a_1)\right) & b_1 R_{12} & \cdots & b_1 R_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & & b_{n-1}R_{n-1,n} \\ b_n R_{n1} & \cdots & & 1 - b_n\left(c_n f''(a_n)a_n + c_n f'(a_n)\right) \end{bmatrix}$$

Then we can choose any b_i in the set $\left(0, \min\left\{\frac{1}{\bar{c}_i}; \min_k \frac{1}{R_{ki}}\right\}\right)$ such that:

$$\begin{aligned} \left|\left[\mathbf{I}-\mathbf{B}\varphi'(\mathbf{a})\right]_{ii}\right| = & \quad \left|1-b_i\left(c_i f''(a_i)a_i + c_i f'(a_i)\right)\right| & < 1 \ \ \forall i \\ \left|\left[\mathbf{I}-\mathbf{B}\varphi'(\mathbf{a})\right]_{ij}\right| = & \quad \left|b_i R_{ij}\right| & < 1 \ \ \forall i \neq j \end{aligned}$$

Hence if we use the sup-norm as our distance metric then for a sufficiently small b_i equation (13) is satisfied. Therefore, for a suitably small b_i we have shown that g is a contraction mapping on T^n and as such, there is unique fixed point $\mathbf{a}^*$ in the set T^n which satisfies the equilibrium equation (6). To complete the proof we show there is no equilibrium $a_i^* > \bar{a}$.

Formally, for all possible $\mathbf{a}$, if at least one $a_i > \bar{a}$ then $\mathbf{a}$ is not an equilibrium action profile. We shall prove this by contradiction.

Let us define a' such that:

$$a' \equiv \bar{a} + \varepsilon$$

Assume there is some $a_k \geq a'$ and $a_k \geq a_j$ for all j. Then the equilibrium condition requires:

$$f'(a_k)a_k = \frac{\sum_j R_{kj}a_j}{c_k} \text{ for agent } k$$

Let us subsitute in $f'(\bar{a}) \equiv \max_i \frac{\sum_j R_{ij}}{c_i}$.

$$\begin{aligned} f'(a_k)a_k &= \frac{\sum_j R_{kj}a_j}{c_k} \text{ for agent } k \\ &\leq \max_i \frac{\sum_j R_{ij}a_j}{c_k} \\ &\leq f'(\bar{a})a_k \end{aligned} \tag{14}$$

Recalling that $f'(a)$ is increasing in a, then $f'(\bar{a})$ must be strictly less than $f'(a')$, therefore inequality (14) cannot hold. Thereby proving that for all possible $\mathbf{a}$, if at least one $a_i > \bar{a}$ then $\mathbf{a}$ is not an equilibrium action profile. A symmetric proof shows that if at least one $a_i < \underline{a}$ and $\mathbf{a} \in R^n_{>0}$ then $\mathbf{a}$ is not an equilibrium action profile.

This completes the proof the centrality-stable $\mathbf{a} \in R^n_{>0}$ is unique. $\square$

A.2. **Proof of Proposition 2.** Under the assumptions in section 2.1; if individual i is an unconnected clone of agent j and has a higher cost parameter $(c_i > c_j)$, then individual i will choose a strictly lower effort $(a_i < a_j)$ in the unique Lindahl equilibrium.

Proof. Recall that the equilibrium condition $J(a)a = 0$ implies:

$$\sum_k R_{ik} a_k = c_i f'(a_i) a_i \ \ \forall i \tag{15}$$

Next, if we assume that ith and jth row of the benefit matrix are identical and there are no reciprocal connections, then the total benefits from the network must be the same for both individuals.[13] Inserting this into equation (15):

$$\sum_k R_{ik} a_k = \sum_k R_{jk} a_k \Rightarrow c_i f'(a_i) a_i = c_j f'(a_j) a_j \tag{16}$$

Without loss of generality assume $c_i < c_j$. Then using equation (16):

$$\frac{f'(a_i) a_i}{f'(a_j) a_j} = \frac{c_j}{c_i} > 1 \tag{17}$$

Recalling that $f(.)$ is strictly convex, therefore it follows that $a_i > a_j$ if and only if $c_i < c_j$. □

A.3. **Proof of Proposition 3.** Under the assumptions in section 2.1, if the elasticity of total costs is decreasing in effort and individual i is an unconnected clone of agent j with a higher cost parameter $(c_i > c_j)$, then agent i will be better off $(U_i(\mathbf{a}) > U_j(\mathbf{a}))$ in the unique Lindahl equilibrium

Proof. Recalling the original utility function $U_i = \sum_k R_{ik} a_k - c_i f(a_i)$ and without loss of generality assume $c_i < c_j$, then i is worse off than j if and only if:

$$\begin{aligned} U_i &< U_j \\ \sum_k R_{ik} a_k - c_i f(a_i) &< \sum_k R_{jk} a_k - c_j f(a_j) \end{aligned}$$

Since by the assumption that two individuals receive the same benefit through the network and there are no reciprocal links we know that $\sum_k R_{ik} a_k = \sum_k R_{jk} a_k$then:

$$\begin{aligned} U_i &< U_j \\ -c_i f(a_i) &< -c_j f(a_j) \end{aligned} \tag{18}$$

[13]The condition that ith and jth row of the benefit matrix are identical and there are no reciprocal connections explicitly requires $R_{ik} = R_{jk}$ for all $k \neq i$ or j and $R_{ij} = R_{ji} = 0$

Using equation (17) we know that[14]:

$$c_i = \frac{c_j f'(a_j)a_j}{f'(a_i)a_i} \tag{19}$$

Sub in equation (19) for c_i in equation (18):

$$\begin{aligned} \frac{c_j f'(a_j)a_j}{f'(a_i)a_i} f(a_i) &> c_j f(a_j) \\ \frac{f'(a_j)a_j}{f(a_j)} &> \frac{f'(a_i)a_i}{f(a_i)} \end{aligned}$$

If we define the elasticity of total cost to be $\epsilon(a_i) = \frac{\partial c_i f(a_i)}{\partial a_i} \frac{a_i}{c_i f(a_i)} = \frac{\partial f(a_i)}{\partial a_i} \frac{a_i}{f(a_i)}$ then:

$$U_i < U_j \Leftrightarrow \epsilon(a_i) < \epsilon(a_j)$$

In the previous proposition we showed a is decreasing in c hence if the elasticity is decreasing in a then the agent with higher cost will be better off. □

A.4. **Proof of Proposition 5.** Under the assumptions in section 2.1, suppose the elasticity of total costs is decreasing or constant in effort and individual i is a connected clone of agent j with higher individual costs ($c_i > c_j$), then agent i will be better off ($U_i(\mathbf{a}) > U_j(\mathbf{a})$) in the unique Lindahl equilibrium.

Proof. Recall that the equilibrium condition $J(a)a = 0$ implies:

$$\sum_k R_{ik} a_k = c_i f'(a_i) a_i \quad \forall i \tag{20}$$

Next, if we assume that ith and jth row of the benefit matrix are identical and have equal reciprocal connections, then the total benefits from the network must be different only by the extent of their respective actions.[15] Therefore, we can write a similar condition to equation (16):

$$\begin{aligned} \sum_{k \neq j} R_{ik} a_k + K a_j &= c_i f'(a_i) a_i \\ \sum_{k \neq i} R_{jk} a_k + K a_i &= c_j f'(a_j) a_j \end{aligned}$$

[14]In the proof we have passed terms across inequalities without worrying about changes in the sign of the inequality since all the terms of the expression must be greater than zero - since we have a convex cost function (so $f'(.) > 0$) and actions must be non-negative.

[15]The condition that ith and jth row of the benefit matrix are identical and there are equal reciprocal connections explicitly requires $R_{ik} = R_{jk}$ for all $k \neq i$ or j and $R_{ij} = R_{ji} = K > 0$

Hence subbing out for $(\sum_{k\neq i} R_{jk}a_k)$ we can combine the two equations into:

$$\begin{aligned}
c_i f'(a_i)a_i - Ka_j &= c_j f'(a_j)a_j - Ka_i \\
c_i f'(a_i)a_i + Ka_i &= c_j f'(a_j)a_j + Ka_j \\
a_i[c_i f'(a_i) + K] &= a_j[c_j f'(a_j) + K] \qquad (21)
\end{aligned}$$

Assume without loss of generality that $c_i < c_j$, then it follows that $a_i > a_j$ due to the strict convexity in $f(.)$. This can be proven by contradiction, assume $a_i \leq a_j$ then for equation (21) to hold, we need:

$$\begin{aligned}
c_i f'(a_i) + K &\geq c_j f'(a_j) + K \\
c_i f'(a_i) &\geq c_j f'(a_j)
\end{aligned}$$

Since we have assumed $c_i < c_j$ it most hold that:

$$f'(a_i) > f'(a_j)$$

However, due to the strict convexity in $f(.)$ then $a_i > a_j$ leading to a contradiction. Therefore agents with higher costs put in lower effort when they have equal benefits from the network and reciprocal connections:

$$c_i < c_j \Leftrightarrow a_i > a_j$$

We are interested in characterising when agent j's utility is greater than agent i's, in the case where $c_i < c_j$, therefore:

$$\begin{aligned}
U_i &< U_j \\
\sum_k R_{ik}a_k - c_i f(a_i) &< \sum_k R_{jk}a_k - c_j f(a_j) \\
\sum_{k\neq j} R_{ik}a_k + Ka_j - c_i f(a_i) &< \sum_{k\neq i} R_{jk}a_k + Ka_i - c_j f(a_j)
\end{aligned}$$

Where in the last equation we have used $R_{ij} = R_{ji} = K > 0$ to define the reciprocal benefits between agents.

$$\begin{aligned}
U_i &< U_j \\
Ka_j + c_j f(a_j) &< Ka_i + c_i f(a_i)
\end{aligned}$$

Using equation (21) we can sub in for c_j:

$$Ka_j + \left[\frac{c_i f'(a_i)a_i + K(a_i - a_j)}{f'(a_j)a_j}\right] f(a_j) < Ka_i + c_i f(a_i)$$

With some rearranging[16]:

$$\begin{aligned} K(a_j - a_i)\left[\frac{f'(a_j)a_j - f(a_j)}{f'(a_j)a_j}\right] &< c_i f(a_i) - \left(\frac{c_i f'(a_i)a_i}{f'(a_j)a_j}\right) f(a_j) \\ K(a_j - a_i)\left[\frac{f'(a_j)a_j - f(a_j)}{f(a_i)f(a_j)}\right] &< c_i\left(\frac{f'(a_j)a_j}{f(a_j)}\right) - c_i\left(\frac{f'(a_i)a_i}{f(a_i)}\right) \end{aligned} \tag{22}$$

Recalling the definition of elasticity of total costs with respect to action is $\epsilon(a_i) = \frac{\partial c_i f(a_i)}{\partial a_i}\frac{a_i}{c_i f(a_i)} = \frac{\partial f(a_i)}{\partial a_i}\frac{a_i}{f(a_i)}$, then we can write equation (22) as:

$$\begin{aligned} K(a_j - a_i)\left[\frac{f'(a_j)a_j - f(a_j)}{f(a_i)f(a_j)}\right] &< c_i\left(\epsilon(a_j) - \epsilon(a_i)\right) \\ K(a_i - a_j)\left[\frac{f'(a_j)a_j - f(a_j)}{f(a_i)f(a_j)}\right] &> c_i\left(\epsilon(a_i) - \epsilon(a_j)\right) \end{aligned}$$

Therefore the LHS is greater than zero since due to the strict convexity in $f(.)$ and the assumption $c_i < c_j$, hence if elasticity of total costs with respect to actions is decreasing in the action, then the individual with lower costs is worse off.

Hence:

$$c_i < c_j \Leftrightarrow a_i > a_j \Rightarrow \begin{cases} U_i < U_j & \text{if } \epsilon(a) \text{is decreasing in } a \end{cases}$$

□

A.5. **Proof of Lemma 1.** An agent's indirect $U_i(c_i,\ a_i(\mathbf{R}, \mathbf{c}))$ is strictly increasing in the equilibrium a_i

Proof. Using the equilibrium condition and the strict concavity assumption on $f(.)$ we can show that $U_i(c_i,\ a_i(\mathbf{R}, \mathbf{c}))$ is strictly increasing in a_i.

Firstly $J(a)a = 0$ implies:

$$\sum_k R_{ik} a_k^* = c_i f'(a_i^*) a_i^* \ \ \forall i \tag{23}$$

[16]Recall that $f(a) > 0$ for all $a > 0$ hence all terms are positive

Recall that:

$$U_i = \sum_k R_{ik} a_k - c_i f(a_i)$$

Substituting in $\sum_k R_{ik} a_k^*$from equation (23) into the utility function:

$$\begin{aligned} U_i(c_i,\ a_i^*) &= c_i f'(a_i^*) a_i^* - c_i f(a_i^*) \\ U_i(c_i,\ a_i^*) &= c_i \left[f'(a_i^*) a_i^* - f(a_i^*) \right] \end{aligned}$$

Due to $f(.)$ being strictly convex, this function is increasing in a_i^*. Formally:

$$\frac{\partial U_i}{\partial a_i^*} = c_i \left[f''(a_i^*) a_i^* \right] > 0$$

□

A.6. **Proof of Proposition 6.** Under the assumptions in section 2.1, an increase in the cost vector $\mathbf{c}' > \mathbf{c}$ will necessarily lead to a decrease in the effort levels for all agents at the the unique Lindahl equilibrium. In particular each agent's change in effort level for a small change in the cost vector is equal to:

$$(\mathbf{a}' - \mathbf{a}) \simeq -\mathbf{D}^{-1}(\mathbf{a})\mathbf{M}(\mathbf{a}) \left(\mathbf{c}' - \mathbf{c} \right) \circ \mathbf{f}'(\mathbf{a}) \circ \mathbf{a}$$

Proof. Consider the equilibrium action for all agents:

$$\sum_k R_{ik} a_k = c_i f'(a_i) a_i \ \ \forall i$$

Define a new parameter $\theta \in [0,1]$, which $\mathbf{c}$ depends on. This implicitly defines a function $\mathbf{a}^* : [0,1] \to \mathbb{R}$, where $\mathbf{a}(\theta)$ is the fixed point when $\mathbf{c} = \mathbf{c}(\theta)$. The parameter θ is used to model how changes in θ affect $\mathbf{c}(\theta)$ and in turn $\mathbf{a}^*(\theta)$.

Writing out the equilibrium condition in the new notation gives us:

$$\begin{aligned} \sum_k R_{ik} a_k(\theta) &= c_i(\theta) f'(a_i(\theta)) a_i(\theta) \ \ \forall i \\ \mathbf{R}\mathbf{a}(\theta) &= \mathbf{c}(\theta) \circ \mathbf{f}'(\mathbf{a}(\theta)) \circ \mathbf{a}(\theta) \end{aligned}$$

Differentiating this function with respect to θ gives:

$$\mathbf{R}\mathbf{a}'(\theta) = \mathbf{c}'(\theta) \circ \mathbf{f}'(\mathbf{a}(\theta)) \circ \mathbf{a}(\theta) + \mathbf{c}(\theta) \circ \mathbf{f}''(\mathbf{a}(\theta)) \circ \mathbf{a}(\theta) \circ \mathbf{a}'(\theta) + \mathbf{c} \circ \mathbf{f}'(\mathbf{a}(\theta)) \circ \mathbf{a}'(\theta)$$

Note when using Hadamard product, $\mathbf{c} \circ \mathbf{f}''(\mathbf{a}(\theta)) \circ \mathbf{a}(\theta) \circ \mathbf{a}'(\theta)$ can be rewritten as $\mathbf{C}[\mathbf{F}'(\mathbf{a}(\theta))\mathbf{A}(\theta)a'(\theta)]$ where $\mathbf{C} = diag(\mathbf{c})$ and $\mathbf{F}''(\mathbf{a}(\theta)) = diag(\mathbf{f}''(\mathbf{a}(\theta)))$ and $\mathbf{A}(\theta) = diag(\mathbf{a}(\theta))$. Therefore, we transform a $n \times 1$ vector into a diagonal $n \times n$ matrix.

Collecting terms in $\mathbf{a}'(\theta)$ and rearranging gives[17]:

$$\begin{aligned} \left(\mathbf{R} - \mathbf{CF}''(\mathbf{a})\mathbf{A} - \mathbf{CF}'(\mathbf{a})\right)\mathbf{a}'(\theta) &= \mathbf{c}'(\theta) \circ \mathbf{f}'(\mathbf{a}) \circ \mathbf{a} \\ \mathbf{a}'(\theta) &= -\left(\mathbf{CF}''(\mathbf{a})\mathbf{A} + \mathbf{CF}'(\mathbf{a}) - \mathbf{R}\right)^{-1}\mathbf{c}'(\theta) \circ \mathbf{f}'(\mathbf{a}) \circ \mathbf{a} \end{aligned} \tag{24}$$

Equation (24) states how the action profile $\mathbf{a}$ changes for changes in the parameter (θ). Define $\mathbf{D} \equiv \mathbf{CF}''(\mathbf{a})\mathbf{A} + \mathbf{CF}'(\mathbf{a})$ and this can be further reduced to:

$$\begin{aligned} \mathbf{a}'(\theta) &= -\left[\mathbf{D} - \mathbf{R}\right]^{-1}\mathbf{c}'(\theta) \circ \mathbf{f}'(\mathbf{a}) \circ \mathbf{a} \\ \mathbf{a}'(\theta) &= -\mathbf{D}^{-1}\left[\mathbf{I} - \mathbf{RD}^{-1}\right]^{-1}\mathbf{c}'(\theta) \circ \mathbf{f}'(\mathbf{a}) \circ \mathbf{a} \end{aligned} \tag{25}$$

If the spectral radius, $\rho(.)$, of $\left[\mathbf{RD^{-1}}\right]$ is less than one, we can write equation (25) as:

$$\mathbf{a}'(\theta) = -\mathbf{D}^{-1}\left[\sum_{k=0}^{\infty}\left(\mathbf{RD^{-1}}\right)^k\right]\mathbf{c}'(\theta) \circ \mathbf{f}'(\mathbf{a}) \circ \mathbf{a} \tag{26}$$

To show $\rho(\mathbf{RD^{-1}}) < 1$, we use lemma 1 in Elliott and Golub (2012a), which shows the spectral radius of $\rho(\mathbf{RE^{-1}}) = 1$, where $\mathbf{E} = \mathbf{CF}'(\mathbf{a})$. By noting

$$\mathbf{D} - \mathbf{E} = \left(\mathbf{CF}''(\mathbf{a})\mathbf{A} + \mathbf{CF}'(\mathbf{a})\right) - \left(\mathbf{CF}'(\mathbf{a})\right) = \mathbf{CF}''(\mathbf{a}) > \mathbf{0}$$

Then the spectral radius:

$$\rho(\mathbf{RD^{-1}}) \quad < \quad \rho(\mathbf{RE^{-1}}) = 1$$

Therefore, the change in the action profile $\mathbf{a}'(\theta)$ can be written in the form of equation (26). Equation (26) shows that an increase in the cost function ($\mathbf{c}'(\theta) \geq 0$ for all elements with at least one strict inequality) will lead to a reduction in the action profile for all agents due to all the terms on the RHS are positive and they are multiplied by a negative. □

[17]For ease of exposition, the term θ has been dropped except from the terms which are differentiated with respect θ

A.7. **Proof of Proposition 7.** Under the assumptions in section 2.1, an increase in $\mathbf{R}'$ from $\mathbf{R}$ such that $\mathbf{R}' > \mathbf{R}$ will necessarily lead to an increase in the effort levels for all agents in the unique Lindahl equilibrium.

Proof. Similar to the proof of Proposition A.6, define a new parameter $\theta \in [0, 1]$, which $\mathbf{R}$ depends on. This implicitly defines a function $\mathbf{a} : [0, 1] \to \mathbb{R}$, where $\mathbf{a}(\theta)$ is the fixed point when $\mathbf{c} = \mathbf{c}(\theta)$. The parameter θ is used to model how changes in θ affect $\mathbf{R}(\theta)$ and in turn $\mathbf{a}(\theta)$. We can show $\mathbf{a}'(\theta)$ is increasing in $\mathbf{R}'(\theta)$. To be precise, recall the original equilibrium condition:

$$\sum_k R_{ik}(\theta)a_k^*(\theta) \quad = \quad c_i f'(a_i^*(\theta))a_i^*(\theta) \ \ \forall i$$

Which we can rewrite as:

$$\mathbf{R}(\theta)\mathbf{a}(\theta) = \mathbf{c} \circ \mathbf{f}'(\mathbf{a}(\theta)) \circ \mathbf{a}(\theta) \tag{27}$$

Following the proof of Proposition A.6, we can differentiate equation (27) and rearrange such that:

$$\begin{aligned} \mathbf{a}'(\theta) &= [\mathbf{D} - \mathbf{R}]^{-1} \mathbf{R}'(\theta)\mathbf{a} \\ \mathbf{a}'(\theta) &= \mathbf{D}\left[\mathbf{I} - \mathbf{R}\mathbf{D}^{-1}\right]^{-1} \mathbf{R}'(\theta)\mathbf{a} \end{aligned}$$

Therefore, since the spectral radius, $\rho(\mathbf{R}\mathbf{D}^{-1}) < 1$, we can write this as:

$$\mathbf{a}'(\theta) = \mathbf{D}\sum_k^{\infty} \left(\mathbf{R}\mathbf{D}^{-1}\right)^k \mathbf{R}'(\theta)\mathbf{a}$$

Hence, an increase in the intensity of links or the introduction of new links $(\mathbf{R}'(\theta) > \mathbf{0})$ in the benefit matrix will lead to an increase in the equilibrium action for all agents. □

A.8. **Proof of Proposition 8.** Under the assumptions in section 2.1, an increase in the cost vector from $\mathbf{c}$ to $\mathbf{c}'$ such that $\mathbf{c} > \mathbf{c}'$ will necessarily lead to a reduction in utility for all agents at the unique Lindahl equilibrium.

Proof. The prof relies on using lemma 1, proposition 6 and proposition 7. First note that the change in the indirect utility, $U_i(c_i,\ a_i(\mathbf{R}, \mathbf{c}))$ can be written as:

$$U_i(c_i,\ a_i(\mathbf{R}, \mathbf{c})) = c_i f'(a_i^*)a_i^* - c_i f(a_i^*)$$

Differentiating with respect to c_k

$$\frac{dU_i}{dc_k} = \mathbf{1_{k=i}}.\left[f'(a_i)a_i - f(a_i)\right] + c_i\left[f''(a_i)a_i\frac{da_i}{dc_k}\right]$$

First consider the case $i \neq k$; the sign $\left(\frac{dU_i}{dc_k}\right) = \text{sign}\left(\frac{da_i}{dc_k}\right)$, since the other terms are greater than zero. Hence following proposition 6, which shows actions are decreasing in costs, then:

$$\left(\frac{dU_i}{dc_k}\right) < 0 \text{ if } k \neq i$$

To show that:

$$\left(\frac{dU_i}{dc_i}\right) < 0$$

Note that a utility function can be transformed with an affine transformation such that an individual's choice of a_i is unchanged from an increase in c_i or an appropriately weighted reduction in the ith row of the benefits matrix.

Formally, consider scaling an individual's cost function by κ, therefore the equilibrium will be such that:

$$\sum_k R_{ik}a_k^* = \kappa c_i f'(a_i^*)a_i^*$$

Further we could replicate the same equilibrium a^* by using $R'_{ik} = \frac{R_{ik}}{\kappa}$, therefore, it follows the change in person $i's$ action from the increase in costs is the same as a reduction in the ith row of the benefit matrix R.

Hence, following proposition 7 which showed actions decreasing in reductions in the benefit matrix and:

$$\frac{dU_i}{dR_{ik}} = c_i\left[f''(a_i)a_i\frac{da_i}{dR_{ik}}\right]$$

Hence, $\text{sign}\left(\frac{dU_i}{dR_{ik}}\right) = \text{sign}\left(\frac{da_i}{dR_{ik}}\right) > 0$. Therefore $\frac{dU_i}{dc_i} < 0$. □

A.9. **Proof of Proposition 9.** Under the assumptions in section 2.1, consider an increase in costs for person k, then the relative impact on person i and person j can be characterised by individual fixed effects $(H_i(a_i)$ and $(H_j(a_j))$, and network effects from person k to person i and j $((M_{ik}(\mathbf{a}))$ and $(M_{jk}(\mathbf{a})))$ in the unique Lindahl equilibrium.

$$\frac{\frac{dU_i}{dc_k}}{\frac{dU_j}{dc_k}} = \frac{H_i(a_i)}{H_j(a_j)} \times \frac{M_{ik}(\mathbf{a})}{M_{jk}(\mathbf{a})} \text{ for } k \neq i \text{ or } j$$

where $H_i(a_i) \equiv \frac{f''(a_i)a_i}{f''(a_i)a_i + f'(a_i)}$

Proof.

$$\begin{aligned} \frac{dU_i}{dc_k} &= \mathbf{1_{k=i}}.\left[f'(a_i)a_i - f(a_i)\right] + c_i\left[f''(a_i)a_i\frac{da_i}{dc_k}\right] \\ &= \frac{f''(a_i)a_i}{f''(a_i)a_i + f'(a_i)}M_{ik}.f'(a_k)a_k \qquad \text{if } i \neq k \end{aligned}$$

Where we have substituted in $\frac{da_i}{dc_k}$ from the earlier proofs. Therefore defining $H_i = \frac{f''(a_i)a_i}{f''(a_i)a_i + f'(a_i)}$ we can denote this as:

$$\frac{dU_i}{dc_k} = H_i M_{ik} f'(a_k)a_k \text{ if } i \neq k$$

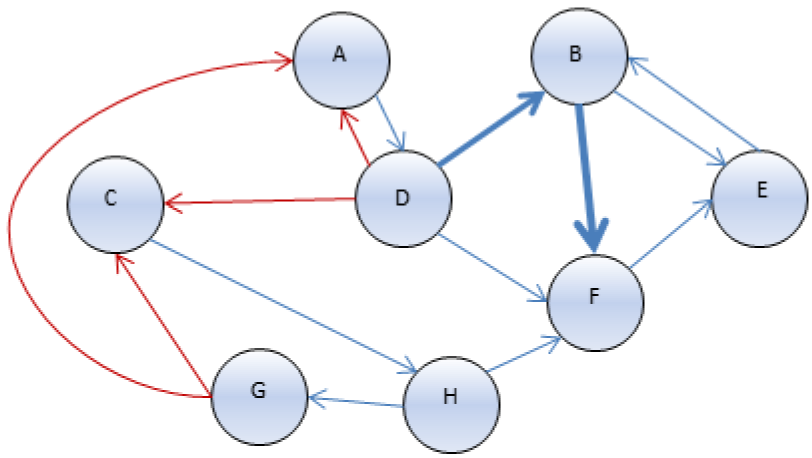
Therefore:

$$\frac{\frac{dU_i}{dc_k}}{\frac{dU_j}{dc_k}} = \frac{H_i}{H_j} \times \frac{M_{ik}}{M_{jk}} \text{ for } k \neq i \text{ or } j$$

□

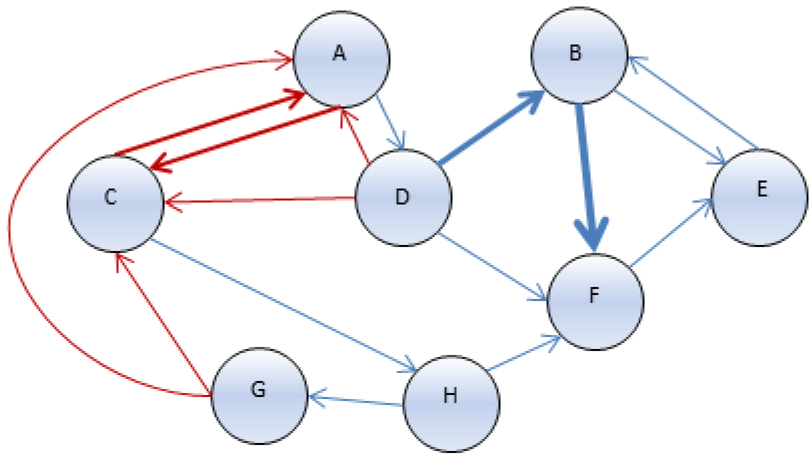
Appendix B. Clones

Figure 5. Agents A and C are unconnected clones in this matrix, since they receive the same marginal benefits from the rest of the network and confer no benefits to each other



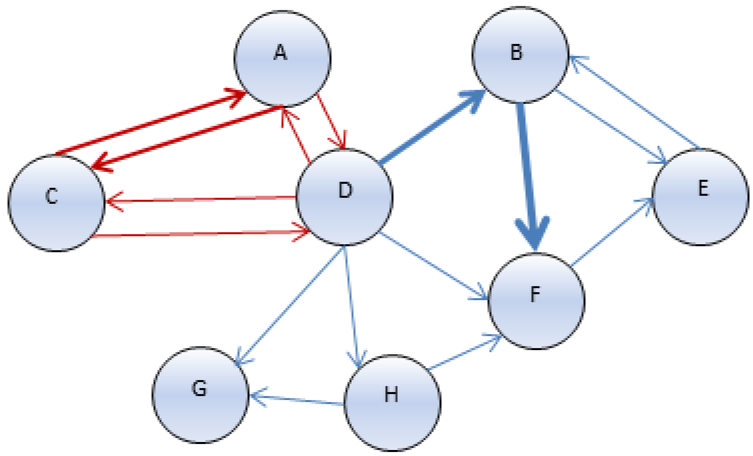
B.1. **Unconnected clones.** The benefits which agents A and C receive are denoted in red for clarity.

FIGURE 6. Agents A and C are connected clones in this matrix, since they receive the same marginal benefits from the rest of the network and confer equal benefits to each other



B.2. **Connected clones.** The benefits which agents A and C receive are denoted in red for clarity.

FIGURE 7. Agents A and C are identical clones in this matrix, since they receive and confer the same marginal benefits from and to the rest of the network.



B.3. **Identical Clones.** Agents A and C must the same cost parameter too (not shown in the figure). The benefits which agents A and C receive and confer are denoted in red for clarity.